An Exactly Solvable Difference Equation that Gives Pure Chaos for a Continuous Range of a Parameter

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A one-dimensional first-order nonlinear difference equation which is an extended version of currently well-studied systems is presented and solved analytically. From the exact solution it is shown that for a continuous range of a parameter of the system (i.e., $0 \leqq k^2 < 1$) nonperiodic solutions behave in a purely chaotic fashion, whereas for $k^2 = 1$ the exact solution converges to a unique attractor.

In order to study chaos produced by one-dimensional nonlinear difference equations, simplest possible systems such as

$$x_{n+1} = a\, x_n\, (1 - x_n) \tag{1}$$

and

$$x_{n+1} = b\, (1 - 2\, |x_n - 1/2|)\,, \tag{2}$$

where $0 \leqq x_n \leqq 1$, $0 \leqq a \leqq 4$ and $0 \leqq b \leqq 1$, are frequently taken up as model systems [1, 2]. We have recently carried out precise investigation of characteristics of Eq. (1) for $a = 4$ and Eq. (2) for $b = 1$ utilizing the fact that they can be solved exactly [3]. In this Letter we report that there exists a difference equation, related to the systems (1) and (2), which can be solvable not only for a particular point but for a continuous range of its parameter and that for the parameter range its nonperiodic solutions are purely chaotic. It is well-known that, for example, (1) produces chaos for a continuous range of its parameter a, namely, $3.5699\ldots < a \leqq 4$. However, Kozak et al. [4] suggested from their computer work that only for $a = 4$ the solutions become purely chaotic supporting a conjecture which had been presented by Grossmann and Thomae [5]. Therefore, it is interesting to see that there actually exists a system that produces pure chaos for a continuous range of its parameter. Another interesting point is that the exact solution of our system suddenly becomes uniformly convergent to an attractor of period 1 at an extreme value of the parameter range that corresponds to the pure-chaos region.

Let us consider a nonlinear difference equation

$$x_{n+1} = \frac{4\, x_n\, (1 - x_n)\, [1 - 4\, k^2\, x_n\, (1 - x_n)]}{1 - [4\, k\, x_n\, (1 - x_n)]^2}\,, \tag{3}$$

where we restrict $0 \leqq k \leqq 1$. Notice that when the parameter $k = 0$, (3) reduces to (1) for $a = 4$. This is a two-parameter system with one of the parameters fixed at 4. The behavior of the rhs of (3) as a function of x_n is shown in Figure 1. Introducing a transformation

$$x_n = (1/2)\, (1 - \mathrm{cn}\, (u_n, k))\,, \tag{4}$$

where $\mathrm{cn}\, (u, k)$ is the Jacobian elliptic function with the argument u and the parameter k, we can rewrite (3) as

$$\mathrm{cn}\, (u_{n+1}) = \frac{-k'^2 + 2\, k'^2\, \mathrm{cn}^2 u_n + k^2\, \mathrm{cn}^4 u_n}{k'^2 + 2\, k^2\, \mathrm{cn}^2 u_n - k^2\, \mathrm{cn}^4 u_n}\,, \tag{5}$$

where $k'^2 = 1 - k^2$ and $\mathrm{cn}\, u = \mathrm{cn}\, (u, k)$. By using the double-argument formula of the elliptic function, we can rewrite (5) as

$$\mathrm{cn}\, (u_{n+1}) = \mathrm{cn}\, (2\, u_n)\,. \tag{6}$$

Once this relation, (6), and the double-argument formula are given, it is straightforward to obtain

$$\mathrm{cn}\, (u_n) = \mathrm{cn}\, (2^n\, u_0)\,. \tag{7}$$

Hence the exact solution of (3) is given as

$$x_n = (1/2)\, [1 - \mathrm{cn}\, \{2^n\, \mathrm{cn}^{-1}\, (1 - 2\, x_0)\}]\,. \tag{8}$$

For $k = 0$, $\mathrm{cn}\, u$ is identical to $\cos u$ and (8) reduces to the exact solution of (1) for $a = 4$ which was given in [3]. For $k = 1$, $\mathrm{cn}\, u$ becomes $\mathrm{sech}\, u$ and

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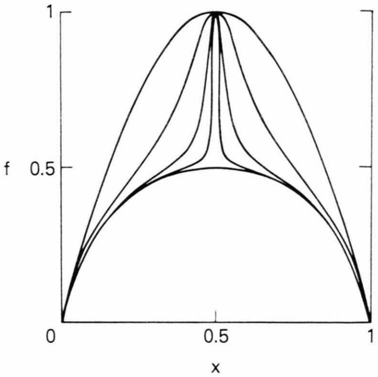

Fig. 1. Graphs of the rhs of (3) as the parameter k^2 is changed. From the top to the bottom the curves correspond to $k^2 = 0$, 0.9, 0.99, 0.999 and 1. For $k^2 = 1$ the peak at $x = 0.5$ disappears.

(8) represents a monotonically converging solution to $x = 1/2$. For this $k = 1$ case the initial value should be restricted as $0 \leqq x_0 \leqq 1/2$, because any value in the region $1/2 < x \leqq 1$ is mapped into the region $0 \leqq x \leqq 1/2$ by the first iteration (see Figure 1).

By imposing a condition on the argument u_n in (6) such that

$$0 \leqq u_n \leqq 2K \quad \text{for all} \quad n \geqq 0\,, \tag{9}$$

where $K = K(k)$ is the complete elliptic integral of the first kind, we see the relationship of (3) to well-studied (2). From (9) we have

$$u_{n+1} = \begin{cases} 2u_n & (0 \leqq u_n \leqq K)\,, \\ 2(2K - u_n) & (K < u_n \leqq 2K) \end{cases} \tag{10}$$

which reduces to (2) for $b = 1$ when scaled by $2K$.

Using (7) the exact solution of (10) is given as

$$u_n = \mathrm{Cn}^{-1}(\mathrm{cn}\,(2^n u_0))\,, \tag{11}$$

where $\mathrm{Cn}^{-1}(.)$ means it is restricted such that $0 \leqq \mathrm{Cn}^{-1}(.) \leqq 2K$.

As we have obtained the exact solution of (3), we can locate the fixed points of period p, which we denote $x^{(p)}$, for any positive integer p. The fixed points of period p is given by the condition, in a new variable $u = 2K\xi$

$$\mathrm{cn}\,(2^p \times 2K\,\xi^{(p)}) = \mathrm{cn}\,(2K\,\xi^{(p)}) \tag{12}$$

and

$$\mathrm{cn}\,(2^q \times 2K\,\xi^{(p)}) \neq \mathrm{cn}\,(2K\,\xi^{(p)}) \tag{13}$$

for any positive integer q less than p.

All the conclusions on the fixed points in ξ-space derived in [3] (Sects. 3 and 4) also hold for the present case. It is evident that our system has fixed points of any period p. Furthermore, we can show that if the initial value in the ξ-space is a rational number, then the solution eventually falls into a periodic fixed points, and if it is an irrational number, then Eq. (8) becomes nonperiodic (see [3]).

In order to show that the nonperiodic solutions of (3) behave in a purely chaotic fashion, we calculate the correlation function defined as

$$C(n) = \frac{\langle (x_n - \langle x_n \rangle)(x - \langle x \rangle)\rangle}{\langle (x - \langle x \rangle)^2 \rangle} = \frac{\langle x_n x \rangle - \langle x_n \rangle \langle x \rangle}{\langle x^2 \rangle - \langle x \rangle^2}\,, \tag{14}$$

where

$$\langle \ldots \rangle = \int_0^1 W(x)(\ldots)\,\mathrm{d}x\,. \tag{15}$$

In (15) the function $W(x)$ denotes the invariant measure which can be readily derived from the exact solution (4) as

$$\begin{aligned} W(x) &= \mathrm{d}u/\mathrm{d}x \\ &= 1/[2K\sqrt{x(1-x)\{k'^2 + k^2(1-2x)^2\}}\,]. \end{aligned} \tag{16}$$

Graphs of $W(x)$ are shown in Fig. 2 for some values of k^2. It is easy to see that

$$\langle x \rangle = \langle x_n \rangle = 1/2\,, \tag{17}$$

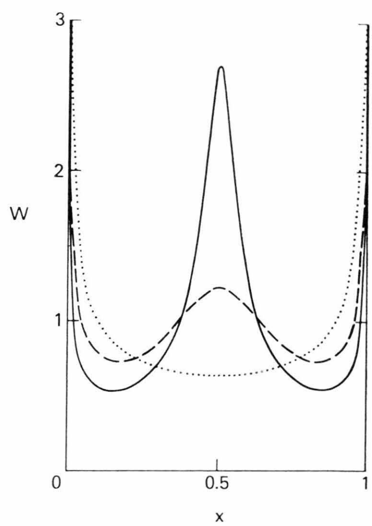

Fig. 2. The invariant measure, (14). Dotted line is for $k^2 = 0$, dashed line $k^2 = 0.9$ and solid line $k^2 = 0.99$. As k^2 approaches 1 the minima become smaller and the peak at $x = 0.5$ grows higher, e.g., for $k^2 = 0.999$, $W(0.5) = 6.52$, and for $k^2 = 0.9999$, $W(0.5) = 16.66$ (not shown here).

whereas

$$\langle x_n x \rangle = 1/4 + I/8K, \tag{18}$$

where

$$I = \int_0^{2K} \mathrm{cn}\,(2^n u)\,\mathrm{cn}\,u\,\mathrm{d}u. \tag{19}$$

For $n=0$ it is obvious that I is a positive finite quantity. Hence $\langle x^2 \rangle$ is a positive finite. For $n \geqq 1$, $I=0$ since $\mathrm{cn}\,u$ is an even function of the period $4K$ and $\mathrm{cn}\,(2^n u)$ has the period of $2K/2^{n-1}$. Consequently, we have $C(0)=1$ and $C(n)=0$ for $n \geqq 1$ which means that nonperiodic solutions of (3) behave in a purely chaotic way for $0 \leqq k^2 < 1$.

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